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THE FREE CATEGORY WITH PRODUCTS ON A MULTIGRAPH

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We describe a 2-dimensional universal property satisfied by the free category-with-products on a multigraph.

Introduction

This paper is part of a series, beginning with [5–8], analysing the syntactical aspects of computer science in terms of free categories with structure, and of presentations of categories with structure. A considerable amount of work has been done on categories with structure by the Sydney school (see for example [1] and the references listed there), and by Lambek [3]. However it is a delicate matter to decide the precise questions to study. The notion of free category-with-structure used by Lambek, while paying appropriate attention to the examples of interest, has not given sufficient attention to 2-categorical aspects. On the other hand, Kelly has analysed well the 2-categorical aspects, but has concentrated attention on the free category-with-structure on a *category* rather than on the more complicated data that arise from the consideration of applications.

The aim of this work is to analyse a simple and fundamental example in detail, namely the free category-with-products on a multigraph, taking into account both the 2-categorical considerations and the appropriate data. The main point is to describe precisely the *correct 2-categorical universal property* satisfied by what is a well-known construction. It is a matter of refining well-known or expected results and concepts. However, we believe that finding the correct universal property is crucial for further developments. We finish by giving a simple coherence theorem which has applications to combinational circuits.

Let me describe briefly the 2-categorical universal property, which seems to be appropriate for many examples of free categories-with-structure. Let $\mathscr{CAT}_{\mathrm{str}}$ be a 2-category of categories-with-structure. Often the data in terms of which such a category is presented is in practice an object $\mathbf{X}$ of some topos $\mathbf{E}$ and there is a forget-

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ful 2-functor $\mathscr{U}:\mathscr{CAT}_{\text{str}}\to\mathscr{CAT}(\mathbf{E})$. Now regard $\mathbf{X}$ as a discrete category in $\mathbf{E}$. The free structured category on $\mathbf{X}$ is a category-with-structure $\mathscr{F}\mathbf{X}$, together with a functor $\Theta:\mathbf{X}\to\mathscr{U}\mathscr{F}\mathbf{X}$, which, for each category $\mathbf{C}$ with structure, induces by composition an equivalence of categories

$$\mathscr{CAT}_{\text{str}}(\mathscr{F}\mathbf{X},\mathbf{C})\simeq\mathscr{CAT}(\mathbf{E})(\mathbf{X},\mathscr{U}\mathbf{C}).$$

The point is that the codomain of $\mathscr{U}$ is $\mathscr{CAT}(\mathbf{E})$, not $\mathbf{E}$, and hence the right-hand side of this equivalence is a category, and not a set. In some interesting cases the universal property is groupoid-enriched rather than category-enriched.

1. Multigraphs

Let $\mathbf{D}$ be the free category on the graph with objects

$$*, 0, 1, 2, 3, \ldots$$

and with $n+1$ arrows from n to $*$ $(n=0,1,2,\ldots)$:

$$d_1, d_2, d_3, \ldots, d_n, c.$$

Then the category **Mgph** of *multigraphs* is $\mathbf{Sets}^{\mathbf{D}}$. If $\mathbf{X}$ is a multigraph, then the elements of $\mathbf{X}_*$ are called *objects*, and the elements of $\mathbf{X}_n$ are called *arrows*. If f is in $\mathbf{X}_n$ and $\mathbf{X}_{d_i}f=X_i$ $(i=1,2,3,\ldots)$ and $\mathbf{X}_cf=Y$, we write

$$f:X_1X_2X_3\ldots X_n\to Y.$$

Let $\mathscr{CAT}_\times$ be the 2-category of categories with finite products, product-preserving functors (in the usual sense) and natural transformations. Then consider a forgetful 2-functor $\mathscr{U}:\mathscr{CAT}_\times\to\mathscr{CAT}(\mathbf{Mgph})$ defined on objects as follows:

- $\mathscr{U}\mathbf{C}_*=\mathbf{C}$,
- $\mathscr{U}\mathbf{C}_n$ for $n\geq 0$ is the category whose objects are $(n+1)$-tuples of arrows of $\mathbf{C}$

  $$p_1:P\to A_1,\ p_2:P\to A_2,\ \ldots,\ p_n:P\to A_n,\ f:P\to B$$

  where P together with p_i $(i=1,2,3,\ldots,n)$ is a product diagram in $\mathbf{C}$. The notion of arrow in $\mathscr{U}\mathbf{C}_n$ between two such objects is the obvious one.
- The effect of $\mathscr{U}\mathbf{C}$ on arrows is the obvious one.

To see the definition of $\mathscr{U}$ on arrows and 2-cells, notice that a product-preserving functor induces a functor in $\mathscr{CAT}(\mathbf{Mgph})$, and that natural transformations between product-preserving functors induce natural transformations.

2. The free category with products on a multigraph

If $\mathbf{X}$ is a multigraph, then the free category-with-products $\mathscr{F}\mathbf{X}$ on $\mathbf{X}$ (in the sense of the introduction) is formed as follows. Objects are words or strings in the objects of $\mathbf{X}$. For each object X of $\mathbf{X}$ consider a sequence of variables $x_1, x_2, x_3, \ldots$ of type X. Then arrows in $\mathscr{F}\mathbf{X}$ are strings of symbols, including commas and brackets, and are defined inductively by:

(i) the variable x_i of type X is an arrow in $\mathscr{F}\mathbf{X}$ of codomain X and domain any word with at least i occurrences of X;

(ii) If U is an object of $\mathscr{F}\mathbf{X}$, while $X_1, X_2, \ldots, X_n$ are objects of $\mathbf{X}$, and $\alpha_i : U \to X_i$ are arrows of $\mathscr{F}\mathbf{X}$ ($i = 1, 2, 3, \ldots, n$), then the string (*including the commas*)

$$\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$$

is an arrow of $\mathscr{F}\mathbf{X}$ from U to $X_1 X_2 \ldots X_n$;

(iii) If $f : X_1 X_2 \ldots X_n \to Y$ is an arrow of $\mathbf{X}$ and $\alpha : U \to X_1 X_2 \ldots X_n$ is an arrow of $\mathscr{F}\mathbf{X}$, then the string (*including the brackets*)

$$f(\alpha)$$

is an arrow of $\mathscr{F}\mathbf{X}$ from U to Y.

In short, arrows are (tuples of) terms constructed out of the arrows of $\mathbf{X}$ regarded as function symbols. It is clear that the method of construction of an arrow can be reconstructed from its form, and its domain and codomain.

Note. To simplify notation we intend to work loosely with variables. Sometimes x_i will mean the ith variable of type X, and at other times it will mean the appropriate variable of type X_i.

Composition $\beta \circ \alpha$ of arrows α, β in $\mathscr{F}\mathbf{X}$ is defined inductively (on the length of β) as follows:

(i) $x_i \circ \alpha_1, \alpha_2, \ldots, \alpha_n = \alpha_i$ if x_i is the variable corresponding to the codomain of α_i;

(ii) $\beta_1, \beta_2, \ldots, \beta_n \circ \alpha = \beta_1 \circ \alpha, \beta_2 \circ \alpha, \ldots, \beta_n \circ \alpha$;

(iii) $f(\beta) \circ \alpha = f(\beta \circ \alpha)$ if f is an arrow of $\mathbf{X}$.

In short, composition is substitution of terms.

The identity of $X_1 X_2 \ldots X_n$ is $x_1, x_2, \ldots, x_n$. The associativity of composition follows by a straightforward inductive argument.

To see that $\mathscr{F}\mathbf{X}$ has finite products, notice that the arrows with codomain $X_1 X_2 \ldots X_n$ are n-tuples of arrows (in a unique way), and part (i) of the definition of composition ensures that the object $X_1 X_2 \ldots X_n$ with projections $x_1, x_2, \ldots, x_n$ is a product diagram in $\mathscr{F}\mathbf{X}$.

Note. In $\mathscr{F}\mathbf{X}$, product is a strictly associative operation.

The functor $\Theta : \mathbf{X} \to \mathscr{U}\mathscr{F}\mathbf{X}$ is defined as follows:

$$\Theta_* X = X,$$

$$\Theta_n(f : X_1 X_2 \ldots X_n \to Y) =$$

$$x_i : X_1 \dots X_n \to X_i (i = 1, 2, \dots, n),\ f(x_1, \dots, x_n) : X_1 \dots X_n \to Y.$$

It remains to check that composition with Θ induces, by composition, an equivalence of categories

$$\mathscr{CAT}_\times(\mathscr{F}\mathbf{X}, \mathbf{C}) \simeq \mathscr{CAT}(\mathbf{Mgph})(\mathbf{X}, \mathscr{U}\mathbf{C}).$$

Let us first prove that if $\mathbf{C}$ is a category with products, then given a functor $\Gamma : \mathbf{X} \to \mathscr{U}\mathbf{C}$ there is a product-preserving functor $\tilde{\Gamma} : \mathscr{F}\mathbf{X} \to \mathbf{C}$ extending Γ in the sense that $\mathscr{U}\tilde{\Gamma}.\Theta$ is isomorphic to Γ.

Such a functor $\tilde{\Gamma}$ is defined inductively as follows. For each n-tuple of objects $X_1, X_2, \dots, X_n$ of $\mathbf{X}$ choose a product diagram in $\mathbf{C}$

$$p_i : \Gamma X_1 \times \Gamma X_2 \times \dots \times \Gamma X_n \to \Gamma X_i \quad (i = 1, 2, \dots, n),$$

and take $\tilde{\Gamma}(X_1 X_2 \dots X_n)$ to be $\Gamma X_1 \times \Gamma X_2 \times \dots \times \Gamma X_n$. If $f : X_1 X_2 \dots X_n \to Y$ is an arrow of $\mathbf{X}$, then the image of f under Γ is an arrow

$$\Gamma f : \Gamma_f \to \Gamma Y$$

together with specified projections from Γ_f to ΓX_i $(i = 1, 2, 3, \dots, n)$, where Γ_f (with the specified projections) is a product in $\mathbf{C}$ of $\Gamma X_1, \Gamma X_2, \dots, \Gamma X_n$. Let

$$\varrho_f : \Gamma X_1 \times \dots \times \Gamma X_n \to \Gamma_f$$

be the unique isomorphism between product diagrams. Then

(i) $\tilde{\Gamma}(x_i : X_1 X_2 \dots X_n \to X_i) = p_i : \Gamma X_1 \times \Gamma X_2 \times \dots \times \Gamma X_n \to \Gamma X_i$,

(ii) $\tilde{\Gamma}(\alpha_1, \alpha_2, \dots, \alpha_n) : \tilde{\Gamma} U \to \tilde{\Gamma}(X_1 X_2 \dots X_n)$
$= (\tilde{\Gamma}\alpha_1, \tilde{\Gamma}\alpha_2, \dots, \tilde{\Gamma}\alpha_n) : \tilde{\Gamma} U \to \Gamma X_1 \times \Gamma X_2 \times \dots \times \Gamma X_n$,

(iii) $\tilde{\Gamma} f(\alpha) = \Gamma f \circ \varrho_f \circ \tilde{\Gamma}\alpha : \tilde{\Gamma} U \to \Gamma X_1 \times \dots \times \Gamma X_n \to \Gamma_f \to \Gamma Y$.

It is a straightforward inductive argument that $\tilde{\Gamma}$ preserves substitution, and hence is a functor. Immediate from the definition is the fact that $\tilde{\Gamma}$ preserves products of the basic types, and hence products in general.

Notice now that $\mathscr{U}\tilde{\Gamma}.\Theta$ is given by

$$\mathscr{U}\tilde{\Gamma}.\Theta_* X = \Gamma X,$$

$$\mathscr{U}\tilde{\Gamma}.\Theta_n(f : X_1 X_2 \dots X_n \to Y) = \Gamma f \circ \varrho_f.$$

It is easy to see that ϱ_f, $f \in \mathbf{X}$, is a natural isomorphism between Γ and $\mathscr{U}\tilde{\Gamma}.\Theta$.

Finally, we check that composition with Θ is fully faithful. Consider two product-preserving functors $\Phi, \Psi : \mathscr{U}\mathscr{F}\mathbf{X} \to \mathbf{C}$. A natural transformation from Φ to Ψ, and a 2-cell from $\mathscr{U}\Phi.\Theta$ to $\mathscr{U}\Psi.\Theta$ both amount to a family of arrows, $\lambda_X : \Phi X \to \Psi X$, of $\mathbf{C}$ indexed by objects of $\mathbf{X}$, and satisfying, for each $f : X_1 \dots X_n \to Y$ of $\mathbf{X}$,

$$\lambda Y \circ \Phi f(x_1, \dots, x_n) = \Psi f \circ \lambda X_1 \times \dots \times \lambda X_n.$$

3. Two consequences

The first consequence we will describe of the above analysis is the well-known fact (see for example [4]) that every category with products is equivalent to a category with strictly associative products.

Consider a category **C** with products. The identity $1_{\mathbf{C}} : \mathscr{U}\mathbf{C} \to \mathscr{U}\mathbf{C}$ in **Mgph** induces a product-preserving functor $\Lambda : \mathscr{F}\mathscr{U}\mathbf{C} \to \mathbf{C}$. Factorize this functor into a bijective-on-objects functor $\Lambda_1 : \mathscr{F}\mathscr{U}\mathbf{C} \to \tilde{\mathbf{C}}$ followed by a fully-faithful functor $\Lambda_2 : \tilde{\mathbf{C}} \to \mathbf{C}$. I claim that Λ_2 is an equivalence, and that $\tilde{\mathbf{C}}$ has strictly associative products. That Λ_2 is an equivalence follows immediately from the fact that Λ is clearly surjective on objects. That $\tilde{\mathbf{C}}$ has strictly associative products follows from the fact that $\mathscr{F}\mathscr{U}\mathbf{C}$ has, and that Λ_1 preserves products and is bijective on objects. Note also, by the way, that Λ_1 is full, so that $\tilde{\mathbf{C}}$ is obtained from $\mathscr{F}\mathscr{U}\mathbf{C}$ by introducing some equations.

The second consequence is a simple coherence theorem, which is closely related to the work of Johnson [2] on pasting theorems in 2-categories, and which forms the basis for the definition of combinational circuits in [8].

Consider a finite multigraph **X**. An object X of **X** is called an *input* object if X does not occur as the codomain of any arrow in **X**. Consider the relation between objects of **X**, written as $X_1 \lhd X_2$, and defined by $X_1 \lhd X_2$ if X_1 occurs in the domain of an arrow f and X_2 is the codomain of f. Consider the transitive closure of this relation, also denoted $X_1 \lhd X_2$. A *loop* in **X** is an object X such that $X \lhd X$.

Definition. A multigraph **X** is called *well formed* if
- Each object is the codomain of at most one arrow,
- If $f : X_1 X_2 \ldots X_n \to Y$ is an arrow of **X** and $X_i = X_j$, then $i = j$.

Proposition. *Consider a finite multigraph* **X**. *If* **X** *is well formed and loop free, then for each object Y in* **X** *there is exactly one arrow in* $\mathscr{F}\mathbf{X}$ *from the product of the input objects to Y.*

Proof. Suppose **X** is well formed and loop free, and that $X_1, X_2, \ldots, X_k$ are the input objects. Then we may define inductively the *depth* $d(X)$ of an object X as follows:

(i) the depth of an input object is zero,

(ii) if $f : Y_1 Y_2 \ldots Y_l \to Y$ is an arrow of **X**, then $d(Y) = \max_{i=1,2,\ldots,l} d(Y_i) + 1$.

The definition is unambiguous since each object is the codomain of at most one arrow. Further, since **X** is loop free and finite, each object is assigned a depth by this prescription (just work backwards from the object). Now we may define an arrow from $X_1 X_2 \ldots X_k$ to Y inductively on the depth of Y. If the depth of Y is zero, take the projection. If the depth of Y is greater than zero and $f : Y_1 \ldots Y_l \to Y$ is an arrow of **X**, then take the arrow from $X_1 X_2 \ldots X_k$ to be $f(\alpha_1, \alpha_2, \ldots, \alpha_l)$ where α_i is the arrow from $X_1 X_2 \ldots X_k$ to Y_i. The fact that there is only one arrow from $X_1 \ldots X_k$

to Y may again be seen by considering the depth of Y. Inspecting the description of $\mathscr{F}\mathbf{X}$, arrows with codomain Y arise only from variables, and from arrows in $\mathbf{X}$ with codomain Y. Hence when Y has depth zero, the only arrow from $X_1X_2...X_k$ to Y is the projection. When Y has depth greater than zero, the arrows from $X_1...X_k$ to Y must be of the form $f(\alpha_1,\dots,\alpha_n)$ where $f: Y_1...Y_n \to Y$ is in $\mathbf{X}$ and α_i is an arrow in $\mathscr{F}\mathbf{X}$ from $X_1...X_k$ to Y_i to Y_i. Argument by induction on the depth shows that there is only one such arrow. □

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References

[1] R. Blackwell, G.M. Kelly and A.J. Power, Two-dimensional monad theory, J. Pure Appl. Algebra 59 (1989) 1-41.
[2] M. Johnson, Pasting diagrams in n-categories with applications to coherence theorems and categories of paths, PhD thesis, University of Sydney, 1988.
[3] J. Lambek and P.J. Scott, Introduction to Higher Order Categorical Logic (Cambridge University Press, Cambridge, 1986).
[4] A.J. Power, A general coherence result, J. Pure Appl. Algebra 57 (1989) 165-173.
[5] R.F.C. Walters, Categories and computer science, 21 lectures, Pure Mathematics Department, University of Sydney, 1988.
[6] R.F.C. Walters, Data types in distributive categories, Bull. Austral. Math. Soc. 40 (1989) 79-82.
[7] R.F.C. Walters, A note on context-free languages, J. Pure Appl. Algebra, 62 (1989) 199-203, this issue.
[8] R.F.C. Walters, A categorical analysis of digital systems, in preparation.